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On Polynomial Interpolation with Mixed Conditions

A. CLAUSING

Institut für Mathematische Statistik, Westfälische Wilhems-Universität, 4400 Münster, West Germany

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1. INTRODUCTION AND PRELIMINARIES

In the following we are concerned with an interpolation problem which is closely related to the classical theorem of G. Pólya on two-point interpolation [3].

We recall briefly the content of this theorem:

Let an interval $[a, b]$ and n linear functionals $\delta_1,..., \delta_n$ be given which are of the form $y_x^{(j)}: f \mapsto f^{(j)}(x)$, with $x = a$ or $x = b$ and $0 \leqslant j \leqslant n-1$. If these are considered as functionals on the space P_{n-1} of real polynomials of degree at most $n-1$, then they are linearly independent if and only if the following condition, now called Pólya's condition, is fulfilled:

For $i = 0,..., n-1$, at least $i+1$ of the given functionals $y_x^{(j)}$ have an order j with $0 \leqslant j \leqslant i$.

Equivalent to $\delta_1,..., \delta_n$ being linearly independent is the existence of a basis of polynomials $A_1,..., A_n$ in P_{n-1} which are biorthonormal to the given functionals: $\delta_i(A_j) = \delta_{ij}$ $(i, j = 1,..., n)$. It is probably well known that none of these polynomials has a zero in the open interval (a, b), so that as a function on $[a, b]$ each of them has a unique sign. The question of how one could read these signs immediately from the δ_i's gave rise to the present paper.

It is a natural approach at this question to replace one of the above "pure" functionals $y_x^{(j)}$ by a "mixed" condition of the form $(1-\alpha) y_x^{(j)} \pm \alpha y_x^{(j+1)}$, and to see what happens if α runs through $[0, 1]$. In this way one can replace $y_x^{(j)}$ by $y_x^{(j+1)}$, and similarly $y_a^{(j)}$ by $y_b^{(j)}$, without losing control on the signs of the basic polynomials, and eventually one arrives at the set $\{y_a^{(i)} \mid 0 \leqslant i \leqslant n-1\}$ whose basic polynomials all have positive sign. By counting all sign changes one obtains the signs of the original polynomials.

This argument is based on an interpolation problem with only one mixed condition, but two-point interpolation with several mixed conditions of the form $\alpha y_x^{(i)} + \beta y_z^{(j)}$ $(x, z \in \{a, b\}, |i-j| \leqslant 1)$ can be treated almost as easily.

This will be done in Section 2 of the paper.

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In their survey article [1] on Hermite–Birkhoff interpolation, Karlin and Karon mentioned the case of mixed conditions as an open question. Theorem 2.3 below may be considered as a reply to that question, if only in a rather restricted situation.

In the other two sections of the paper, the basic polynomials for the interpolation problem of Section 2 are discussed. An estimate on the number of their zeros is given and their signs are being determined for some cases in which they exist.

Throughout the paper, we fix a finite interval $[a, b]$. Let $D = \{\delta_1, ..., \delta_n\}$ be a set of n linear functionals of the form $\delta_i = \sum_{k=1}^{m} \alpha_k y_{x_k}^{(j_k)}$ with $0 \leqslant j_1 \leqslant \cdots \leqslant j_m \leqslant n-1$ and $\alpha_k \neq 0$, $x_k \in \{a, b\}$ for all k. Call j_1 the order of such a functional. When speaking of linear independence of D, we take its elements as linear functionals on P_{n-1}. Linear independence thus means that for all $f \in P_{n-1}$, $f \not\equiv 0$, there is at least one i, $1 \leqslant i \leqslant n$, with $\delta_i(f) \neq 0$ or, equivalently, that the determinant $|D| = |\delta_i(x^{j-1})|_{i,j=1,...,n}$ does not vanish.

Let M_j be the number of functionals in D of order at most j. The proof of the following lemma is almost identical to that of Pólya for the pure case, so we refer the reader to [3] or [4] for details.

Lemma 1.1. *If D is linearly independent, then Pólya's condition holds:*

$$M_j \geqslant j+1 \qquad (j = 0, ..., n-1). \quad \blacksquare$$

Evidently, in the mixed case the Pólya condition is no longer sufficient for linear independence. As an example, consider $D = \{y_0, y_1 + \alpha y_1', y_0' + \beta y_1'\}$. The determinant of the matrix $(\delta_i(x^{j-1}))_{i,j=1,2,3}$ is $|D| = -1 - 2\alpha + \beta$. A very crude sufficient condition for $|D| \neq 0$ is $\alpha \geqslant 0$ and $\beta \leqslant 0$. It is by this kind of conditions that we can ensure the linear independence of the sets D in the next section without having to compute a determinant.

2. A Sufficient Condition for Linear Independence

Let g be a real-valued C^1-function on $[a, b]$, let z be the number of zeros of g in the open interval (a, b) and z' the number of zeros of g' in (a, b).

Consider the following five conditions on g, where $m = 1$ or $m = 2$, $\alpha \geqslant 0$ and $\beta \geqslant 0$:

(1) $g(a) - \alpha g'(a) = 0$,

(2) $g(a) + (-1)^z \alpha g(b) = 0$,

(3) $g(a) - (-1)^{z+m} \alpha g'(b) = 0$,

(4) $g(b) + \beta g'(b) = 0$,

(5) $g(b) + (-1)^{z+m} \beta g'(a) = 0$.

We shall need the following elementary lemma whose proof is straightforward and will be omitted.

LEMMA 2.1. *Let m be either 1 or 2. If $m=1$, suppose that at least one of the above five conditions holds. If $m=2$, suppose that one of the conditions* (1), (2), (3) *and one of* (4), (5) *holds. If, for $m=2$,* (3) *and* (5) *are supposed to hold, we also assume $z=0$. Then*

$$z' \geqslant z+m-1. \quad \blacksquare$$

We remark that for $m=2$, $z \geqslant 1$, conditions (3) and (5) in general do not imply $z' \geqslant z+1$.

Now let us fix a set E of n linear functionals of the form $y_a^{(j)}$ or $y_b^{(j)}$ $(0 \leqslant j \leqslant n-1)$ which fulfils Pólya's condition: $M_j \geqslant j+1$ for $j=0,..., n-1$. Also, put $M_{-1}=0$ and let m_j denote the number of functionals in E which have the order j.

To E we relate a set D of mixed linear functionals in the following way:

(A) If $y_a^{(j)} \in E$, then one and only one of the following functionals is in D, where $\alpha=\alpha(a,j)>0$:

$$y_a^{(j)}, \qquad y_a^{(j)}-\alpha y_a^{(j+1)}, \qquad y_a^{(j)}+\alpha(-1)^{M_{j-1}-j}\, y_b^{(j)},$$
$$y_a^{(j)}+\alpha(-1)^{M_{j-1}-(j+1)}\, y_b^{(j+1)}.$$

(B) If $y_b^{(j)} \in E$, then one and only one of the following functionals is in D, where $\beta=\beta(b,j)>0$:

$$y_b^{(j)}, \qquad y_b^{(j)}+\beta y_b^{(j+1)}, \qquad y_b^{(j)}+\beta(-1)^{M_{j-1}-j}\, y_a^{(j)},$$
$$y_b^{(j)}-\beta(-1)^{M_{j-1}-(j+1)}\, y_a^{(j+1)}.$$

It is convenient to refer to them as pure, lateral, transversal, and diagonal conditions, respectively.

We also ask the following restriction on D:

(C) If $m_j=2$, that is if there are two conditions of order j in D, then they may not both be diagonal unless $M_{j-1}=j$, and if both are transversal, they must not be proportional.

Such a set D obviously meets Pólya's condition. An inequality which is familiar in connection with that condition persists in our setting:

PROPOSITION 2.2. *Let D have properties* (A)–(C). *Let $f \in C^n[a,b]$ fulfil $\delta(f)=0$ for all $\delta \in D$ and denote the number of zeros of $f^{(j)}$ in (a,b) by n_j. Then*

$$n_j \geqslant M_{j-1}-j \qquad (j=0,..., n-1).$$

Proof. We use induction by j. For $j=0$, nothing is to show, so let us suppose that the inequality is true for some j, $0 \leqslant j \leqslant n-2$. If $m_j=0$, then by Rolle's theorem $n_{j+1} \geqslant n_j - 1 \geqslant M_{j-1} - j - 1 = M_j - (j+1)$. If $m_j = 1$, $M_j - (j+1) = M_{j-1} - j$. Hence the induction step is trivial if $n_j = 0$ or $n_j > M_{j-1} - j \geqslant 1$, by Rolle's theorem. If $n_j = M_{j-1} - j$, we can apply Lemma 2.1 to $g = f^{(j)}$, putting $m=1$ and $z = M_{j-1} - j$. We obtain $n_{j+1} \geqslant n_j$, which is clearly sufficient.

If $m_j = 2$ we have $M_j - (j+1) = M_{j-1} - j + 1$, so by Rolle's theorem again there is no problem if $n_j > M_j - (j+1)$. If $n_j = M_j - (j+1)$, we can apply Lemma 2.1 to $g = f^{(j)}$ with $m = 1$ and $z = M_j - (j+1)$, provided that not both order j conditions are transversal. But this case is equivalent to having two pure conditions, by (C), so it can be excluded. If $n_j = M_{j-1} - j$, Lemma 2.1 with $g = f^{(j)}$, $m=2$, and $z = M_{j-1} - j$ applies. This shows $n_{j+1} \geqslant M_j - (j+1)$, and the proof is complete. ∎

Now we are in the position to prove our first theorem.

THEOREM 2.3. *If the set D has properties* (A)–(C), *then it is linearly independent.*

Proof. Let f be a polynomial of degree j, $0 \leqslant j \leqslant n-1$, which fulfils $\delta(f) = 0$ for all $\delta \in D$. It suffices to show $f \equiv 0$. Let us assume the contrary. If there were no mixed conditions of order j in D, then the number N_j of zeros of $f^{(j)}$ in $[a, b]$ would be $N_j = n_j + m_j \geqslant M_{j-1} - j + m_j = M_j - j \geqslant 1$, whereas $N_j = 0$ as $f^{(j)} \equiv c \neq 0$. A lateral or diagonal condition of order j is impossible as $f^{(j+1)} \equiv 0$ and $f^{(j)}(a) \neq 0, f^{(j)}(b) \neq 0$. Hence there should be a transversal condition of order j in D, and by $0 = n_j \geqslant M_{j-1} - j \geqslant 0$ we get $f^{(j)}(a) + \alpha f^{(j)}(b) = c(1+\alpha) = 0$ for some $\alpha > 0$. This contradicts $c \neq 0$, thus our assumption is disproved. ∎

Lemma 2.1 and hence the proof of Theorem 2.3 do not work in the case of two diagonal conditions of an order j with $M_{j-1} - j > 0$. In fact, the conclusion of our theorem is not true in this case. Take $D = \{y_0, y_1, y_0' + \alpha y_1'', y_1' - \beta y_0''\}$, which for $\alpha, \beta > 0$ satisfies our conditions (A) and (B), but not (C), because $M_{j-1} - j = 1$ for $j = 1$. Its associated determinant is $|D| = 12\alpha\beta - 2\alpha - 2\beta - 1$, which can be zero not only for $\alpha > 0$ and $\beta > 0$ but for any prescribed signs of α and β. Therefore the choice of signs in (A) and (B) cannot be made in a way that includes this exceptional case.

3. ZEROS OF THE BASIC POLYNOMIALS

Let $D = \{\delta_1, ..., \delta_n\}$ be a set of functionals having properties (A)–(C) of the last section. Then the system of equations $x^{j-1} = \sum_{i=1}^{n} \delta_i(x^{j-1}) A_i$

$(j=1,\dots,n)$ has a unique set of solutions $A_1,\dots,A_n$ in P_{n-1}, the *basic polynomials* of D. Obviously, they are biorthonormal to the functionals: $\delta_i(A_j)=\delta_{ij}$ $(i,j=1,\dots,n)$.

For studying the eigenvalue problem $y^{(n)}=\lambda y$, $\delta_i(y)=0$ $(i=1,\dots,n)$ as well as for other applications (for instance, in the next section), it would be desirable to know when the basic polynomials have no zeros in (a,b).

While this holds in the pure case, it is by no means true in general, not even in very simple examples like $D=\{y_0+y_1, y_0'\}$, where $A_1(x)=\frac{1}{2}$, $A_2(x)=x-\frac{1}{2}$.

In some mixed cases, however, the following estimate is sufficient to show that the basic polynomials indeed have no zeros in (a,b).

For $i=1,\dots,n$, let j_i be the degree of A_i and S_i the number of conditions in D whose order is at most j_i-1 and who are either transversal, or diagonal, or equal to δ_i. Put $S_i=0$ if $j_i=0$. Let z_i be the number of zeros of A_i in (a,b).

THEOREM 3.1. *For $i=1,\dots,n$ the following estimate holds:*

$$z_i \leqslant S_i-(M_{j_i-1}-j_i).$$

Proof. We may assume $j_i\geqslant 1$. By $V(x)$, resp. $U(x)$, let us denote the number of sign changes, resp. sign constancies, in the sequence $\sigma(x)=(A_i(x), A_i'(x), A_i''(x),\dots,A_i^{(j_i)}(x))$, zeros being discarded. By the Budan–Fourier theorem [2, p. 65], A_i has at most $V(x)-V(y)$ zeros in any half-open interval $(x,y]$. If $\sigma(a)$ contains any zero terms, say, $A_i^{(k)}(a)=A_i^{(k+1)}(a)=\cdots=A_i^{(j)}(a)=0$, $A_i^{(j+1)}\neq 0$, then for $x=a+\varepsilon$ with sufficiently small $\varepsilon>0$ the sequence $\sigma(x)$ contains no zeros, the signs of $A_i^{(m)}(a)$ and $A_i^{(m)}(x)$ coincide for all m, $0\leqslant m\leqslant j_i$ with $A_i^{(m)}(a)\neq 0$, and the terms $A_i^{(k)}(x)$, $A_i^{(k+1)}(x),\dots,A_i^{(j+1)}(x)$ all have the same sign. This is an immediate consequence of the Taylor formula. Likewise, if $\sigma(b)$ contains zero terms, say, $A_i^{(k)}(b)=A_i^{(k+1)}(b)=\cdots=A_i^{(j)}(b)=0$, $A_i^{(j+1)}(b)\neq 0$, then for $y=b-\varepsilon$ with $\varepsilon>0$ sufficiently small, the sequence $\sigma(y)$ contains no zeros, the signs of the non-zero terms in $\sigma(b)$ coincide with their counterparts in $\sigma(y)$, and the sequence $A_i^{(k)}(y), A_i^{(k+1)}(y),\dots,A_i^{(j+1)}(y)$ alternates in sign. Especially, for any pure condition $y_a^{(j)}\in D$, $j\leqslant j_i-1$, $y_a^{(j)}\neq\delta_i$, the terms $A_i^{(j)}(x)$ and $A_i^{(j+1)}(x)$ have equal signs, and for $y_b^{(j)}\in D$, $j\leqslant j_i-1$, $y_b^{(j)}\neq\delta_i$, the signs of $A_i^{(j)}(y)$ and $A_i^{(j+1)}(y)$ are opposite. Also, every lateral condition at a which is unequal δ_i corresponds to an equality of two subsequent signs in $\sigma(x)$ and every lateral condition at b, different from δ_i, corresponds to a sign change in $\sigma(y)$. Hence, if p_i is the number of pure or lateral conditions in D which are different from δ_i and whose order is at most j_i-1, then

$$p_i\leqslant U(x)+V(y).$$

Observing that

$$j_i = U(x) + V(x)$$

and that

$$z_i \leqslant V(x) - V(y)$$

for sufficiently small values of $\varepsilon > 0$ by the Budan–Fourier theorem, we obtain

$$z_i \leqslant (j_i - U(x)) + (U(x) - p_i) = j_i - p_i.$$

But $S_i + p_i = M_{j_i - 1}$ by definition, so the theorem is proved. ∎

The following immediate consequence will be used in the next section.

COROLLARY 3.2. *If D contains no transversal or diagonal condition of order* j, $0 \leqslant j \leqslant n-2$, *then none of the basic polynomials has a zero in* (a, b).

Proof. In that case, $S_i = 0$ for $i = 1,..., n$. ∎

In the classical case, where all conditions are pure, this result can be obtained more easily using a theorem of Schoenberg: If one assumes that $A_i(x) = 0$ for some $x \in (a, b)$, then the conditions $\delta_j \in D, j \neq i$, together with y_x constitute a quasi-Hermite interpolation system, and hence $A_i \equiv 0$ by Theorem 2 of [4], contradicting $\delta_i(A_i) = 1$.

We remark that the estimate $z_i \leqslant V(a) - V(b)$, which seems to be the most natural way to apply the Budan–Fourier theorem, does not lead to a proof of Theorem 3.1, even if $A_i(b) \neq 0$. In the example $D = \{y_0, y_1 + y_0', y_1', y_1''\}$ the polynomial A_1 according to $\delta_1 = y_0$ is $A_1(x) = \frac{1}{4}(3 - (x-1)^3)$. Here, $V(0) - V(1) = 2$, but Theorem 3.1 correctly predicts $z_1 = 0$.

4. SIGNS OF THE BASIC POLYNOMIALS

In the situation of Corollary 3.2, let the sign of the basic polynomial A_i in the interval $[a, b]$ be ε_i.

LEMMA 4.1. *The signs* $\varepsilon_1,..., \varepsilon_n$ *are determined by the underlying set E of pure conditions.*

Proof. For $i = 1,..., n$, the functional $\delta_i \in D$ is of the form $y_{x_i}^{(j_i)} + \alpha_i y_{z_i}^{(k_i)}$, subject to restrictions (A)–(C) in Section 2 and, in particular, with $y_{x_i}^{(j_i)}$ being in E. For $0 \leqslant t \leqslant 1$, consider the set D_t consisting of $\delta_i^t = y_{x_i}^{(j_i)} + t\alpha_i y_{z_i}^{(k_i)}$ $(i = 1,..., n)$. Then $D_0 = E$ and $D_1 = D$, and D_t satisfies the condition of

Corollary 3.2 as D does, so that its fundamental polynomials A_i^t have no zeros in (a, b). Now fix some $x \in (a, b)$. The function $t \mapsto A_i^t(x)$ on $[0, 1]$ is continuous as $A_i^t(x)$ is the solution of a system of linear equations whose coefficients depend continuously on t and whose determinant never vanishes. As $A_i^t(x) \neq 0$ for all $t \in [0, 1]$, the function $t \mapsto \operatorname{sign} A_i^t(x)$ is a constant which clearly equals ε_i. ∎

Now let $E = \{\delta_1, ..., \delta_n\}$ be any set of pure functionals which satisfies Pólya's condition.

We say that $E' = \{\delta_1', ..., \delta_n'\}$ arises from E by replacing the functional $\delta_{i_0} = y_x^{(j)} \in E$ with $y_z^{(k)}$, if $\delta_{i_0}' = y_z^{(k)}$, $\delta_i' = \delta_i$ for $i \neq i_0$, and if E' also fulfils Pólya's condition (in particular, $y_z^{(k)} \notin E$).

LEMMA 4.2. *If in E the functional $\delta_{i_0} = y_a^{(j)}$ is replaced by $y_a^{(j+1)}$, then the signs ε_i and ε_i' of the respective basic polynomials of E and E′ are related by $\varepsilon_{i_0}' = -\varepsilon_{i_0}$, $\varepsilon_i' = \varepsilon_i$ $(i \neq i_0)$. If $y_a^{(n-1)}$ is replaced by $y_b^{(n-1)}$, then $\varepsilon_i' = \varepsilon_i$ for all $i = 1, ..., n$. The same holds if $y_b^{(j)}$ is replaced by $y_b^{(j-1)}$.*

Proof. For the first statement, we consider the set D_t consisting of the functionals δ_i $(i \neq i_0)$ and $(1-t)\, y_a^{(j)} - t y_a^{(j+1)}$ $(0 \leqslant t \leqslant 1)$. Then D_t fulfils (A)–(C) of Section 2, except for the different norming of the first term, which is obviously not relevant. For $t \neq 1$, it follows from Lemma 4.1 that the basic polynomials of D_t all have the same signs as their counterparts in $D_0 = E$. For continuity reasons, this remains true for $t = 1$. But E' differs from D_1 only in that condition $y_a^{(j+1)}$ in E' is replaced by $-y_a^{(j+1)}$ in D_1. This proves the first statement. For the second one, we construct an analogous set D_t using the condition $(1-t)\, y_a^{(n-1)} + t y_b^{(n-1)}$. Note that $M_{j-1} - j = 0$ for $j = n-1$ in this situation, as there is exactly one condition of order $n-1$ in D_t. Clearly, Corollary 3.2 applies to D_t. For the third statement we use $t y_b^{(j-1)} + (1-t)\, y_b^{(j)}$ in a similar way. ∎

To determine the signs of the basic polynomials it is convenient to number the conditions in E as follows:

Let $\delta_1 = y_a^{(j_1)}$, $\delta_2 = y_a^{(j_2)}, ..., \delta_m = y_a^{(j_m)}$ be the conditions at a in increasing order, i.e. $j_i < j_k$ if $i < k$, and let $\delta_{m+1} = y_b^{(j_{m+1})}, ..., \delta_n = y_b^{(j_n)}$ be the conditions at b in decreasing order, i.e., $j_k < j_i$ if $i < k$.

THEOREM 4.3. *Let D satisfy the requirements of Corollary 3.2 and let its elements be indexed according to the above numbering of its underlying set E. Then the signs of its basic polynomials are*

$$\varepsilon_i = (-1)^{i+j_i+1} \qquad \text{for } i = 1, ..., m,$$

$$\varepsilon_i = (-1)^{n+i} \qquad \text{for } i = m+1, ..., n.$$

Proof. We may in fact restrict us to a set of pure conditions by Lemma 4.1. If this set happens to be $\{ y_a^{(i)} \mid 0 \leqslant i \leqslant n-1 \}$, then

$$A_i(x) = \frac{1}{i!} \left(\frac{x-a}{b-a} \right)^i,$$

so that $\varepsilon_i = 1$ for all i in this case. Now the idea is to move first $\delta_n = y_a^{(n-1)}$ to its final position $y_b^{(j_n)}$, then $\delta_{n-1} = y_a^{(n-2)}$ to $y_b^{(j_{n-1})}$, and so on, doing this by repeated application of the replacements described in Lemma 4.2 and at the same time counting all sign changes. Assume that this has been done for $\delta_n, \delta_{n-1}, \ldots, \delta_{k+1}$ for some $k > m$. Then, as $k > m$, there is no condition of order $n-1$, so we may replace $\delta_k = y_a^{(k-1)}$ with, successively, $y_a^{(k)}, y_a^{(k+1)}, \ldots, y_a^{(n-1)}, y_b^{(n-1)}, y_b^{(n-2)}, \ldots, y_b^{(j_k)}$, without violating Pólya's condition. By Lemma 4.2 the signs ε_i, $i \neq k$, remain unchanged during this process, whereas ε_k is changed $n-k$ times, so that finally $\varepsilon_k = (-1)^{n+k}$. After doing this for $k = m+1$, we move $\delta_m = y_a^{(m-1)}$ to $y_a^{(j_m)}$, changing ε_m $m - j_m - 1$ times, and so on, until δ_1 is in its final position. This finishes the proof. ∎

Added in proof. With respect to the particular numbering used in this theorem, it is even true that the polynomials $\varepsilon_1 A_1, \ldots, \varepsilon_n A_n$ form a Descartes system on (a, b). A more general result will be proved in a forthcoming paper.

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